On Metric Sorting for Successive Cancellation List Decoding of Polar Codes

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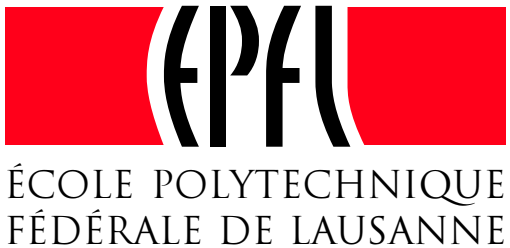

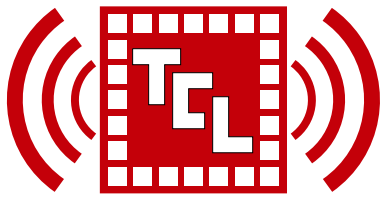

ISCAS 2015, Lisbon

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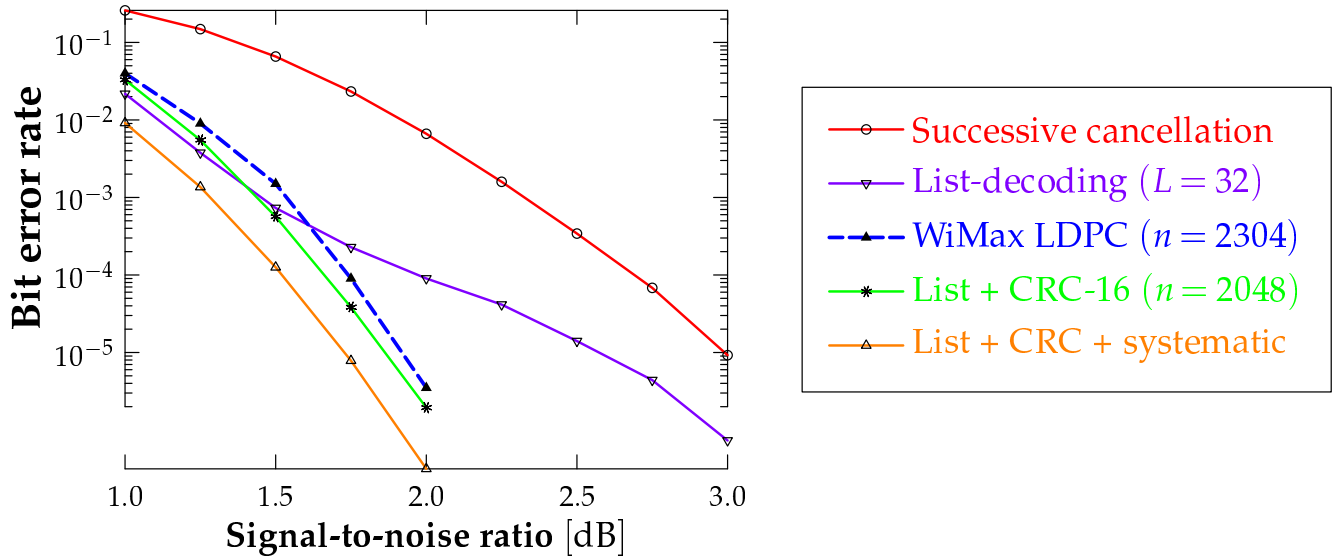

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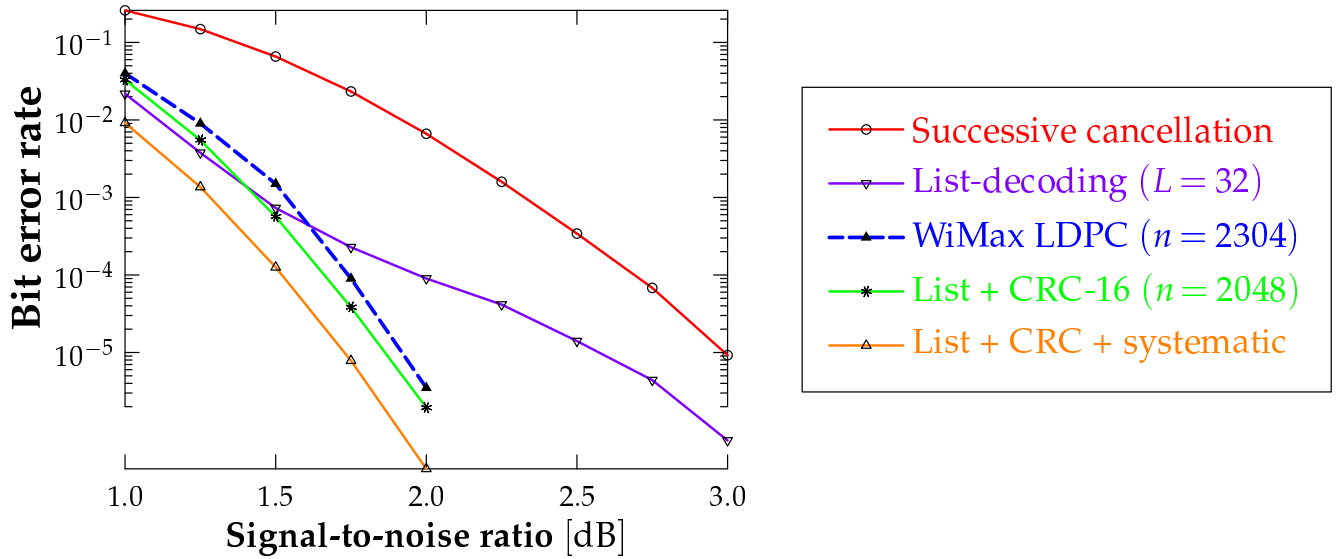

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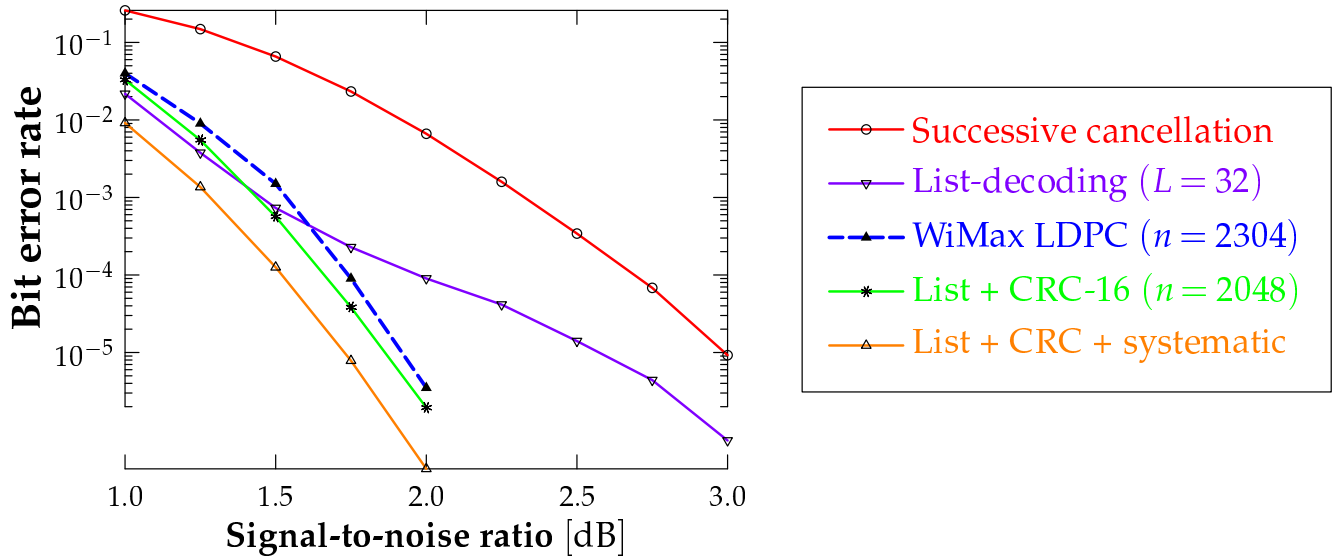

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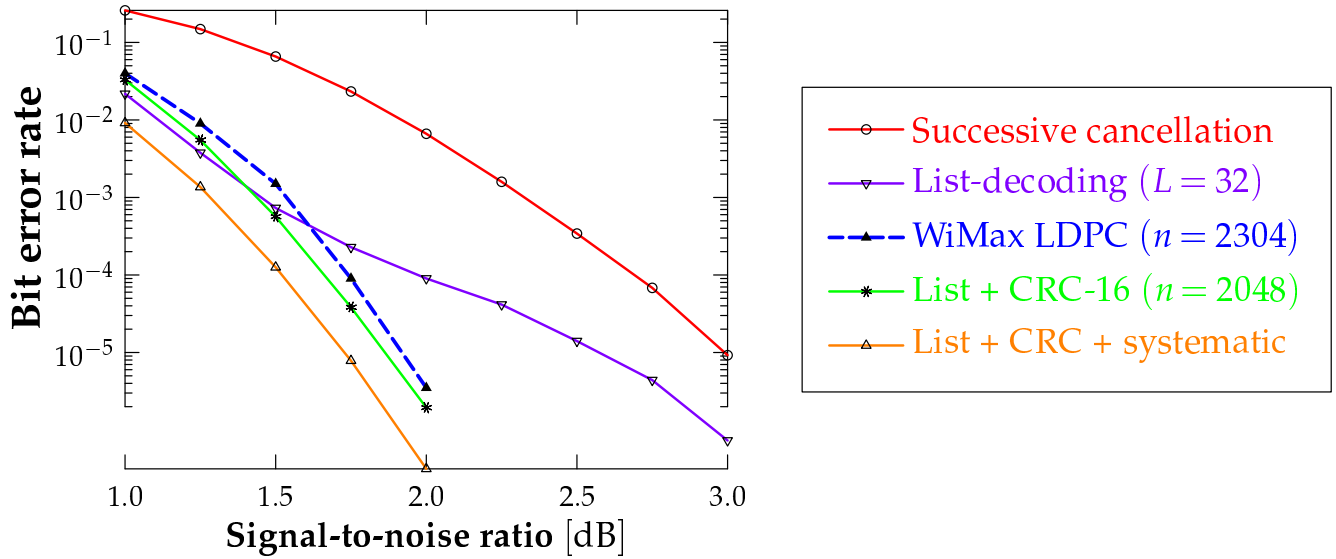

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- Main step of SCL decoding: choose L best paths out of $2L$ possible ones:
  - Each tentative path is associated with a metric
  - The list of $2L$ path metrics is sorted.
  - First L metrics in the sorted list correspond to the surviving paths.

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- The metric sorter is a **crucial** component which quickly becomes the critical path as L is increased.

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Different list size $\rightarrow$ **Different** optimal sorter choice

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$$m_0, m_1, m_2, m_3, \ldots, m_{2L-2}, m_{2L-1}$$

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  - half of the metrics previously existing path metrics which are already sorted,

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- We don't need to sort the entire list.
  It is sufficient to find a sorted list of the L smallest elements of the list.

Radix-$2L$ and Pruned Radix-$2L$ Sorters

- **Radix-$2L$ sorter**

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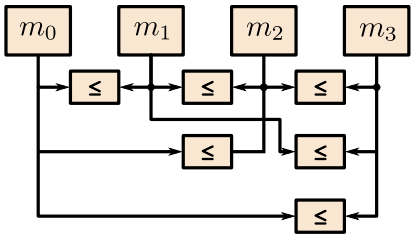

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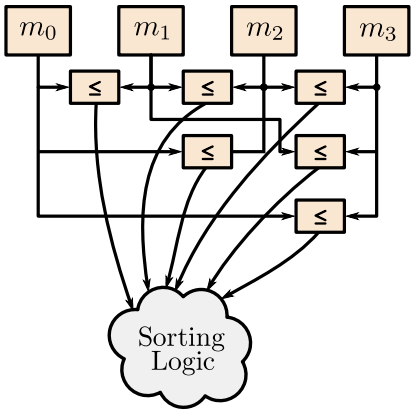

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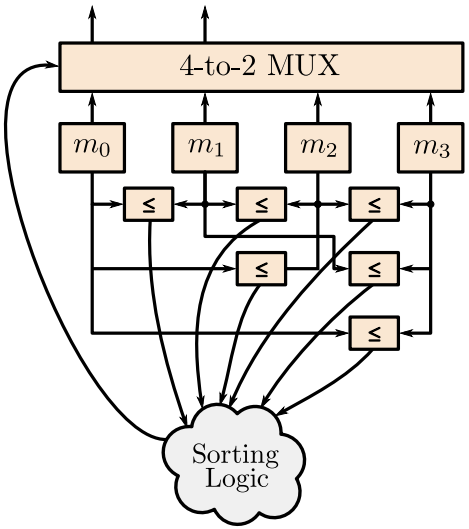

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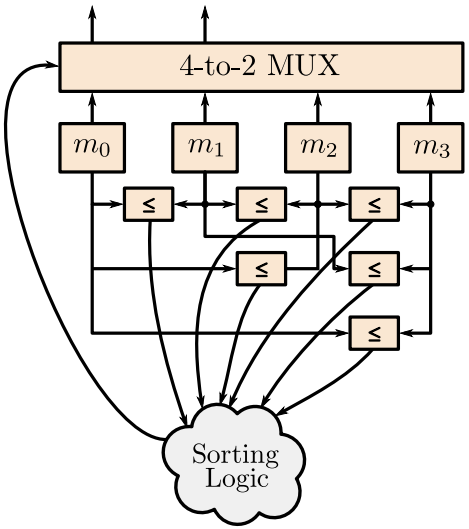

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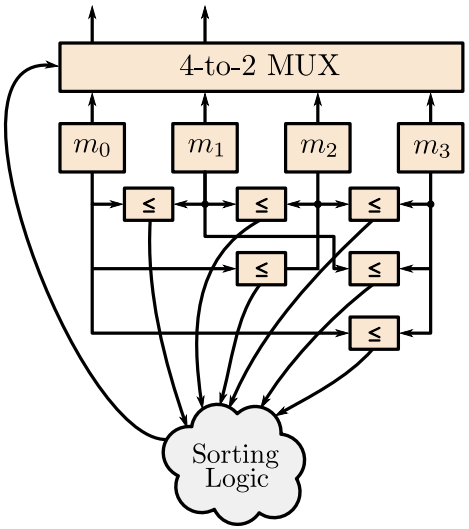

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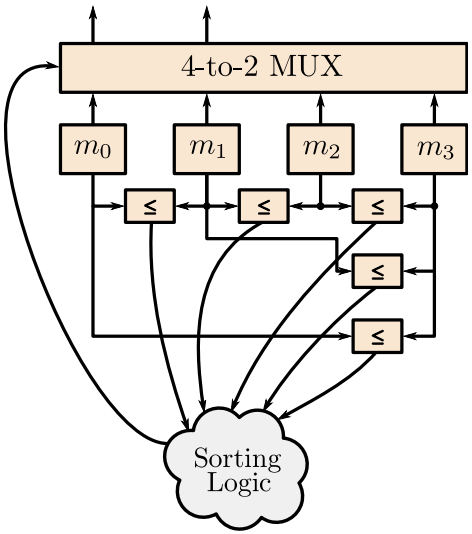

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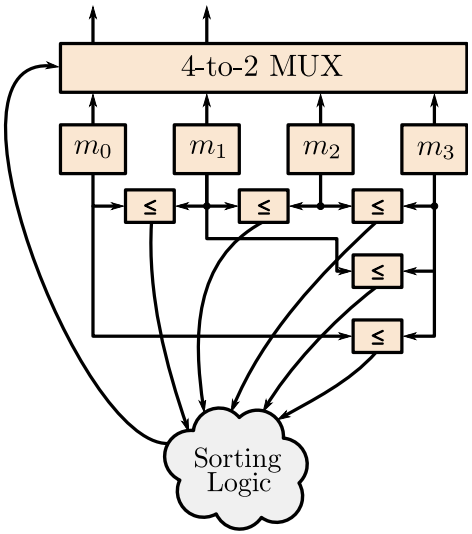

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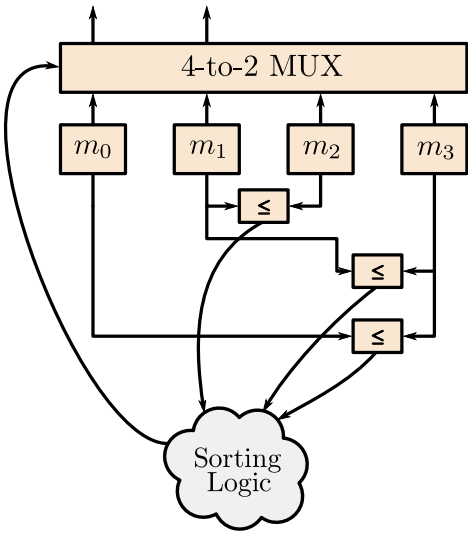

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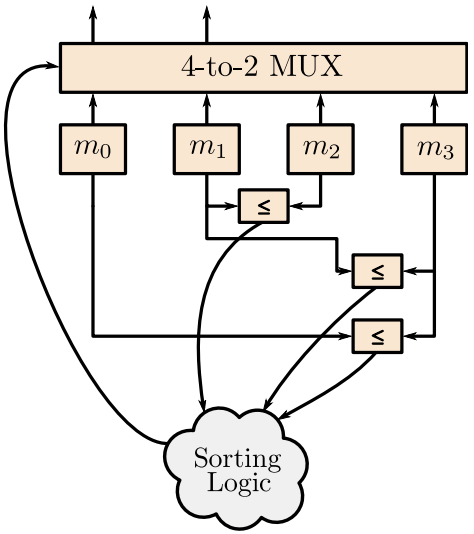

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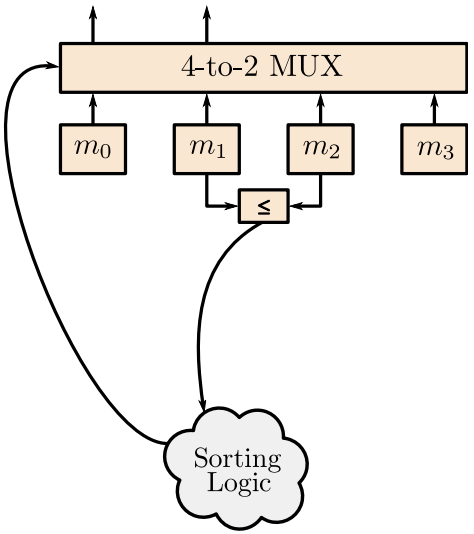

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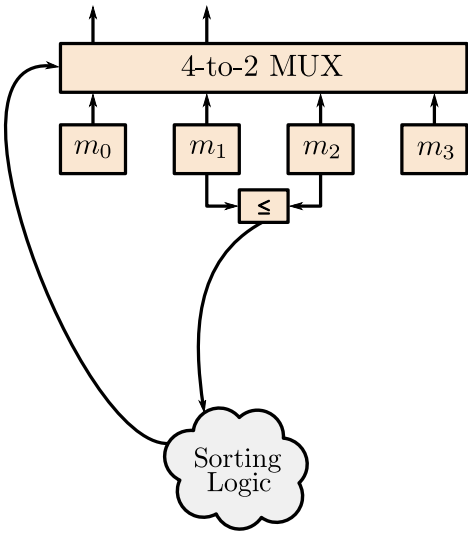

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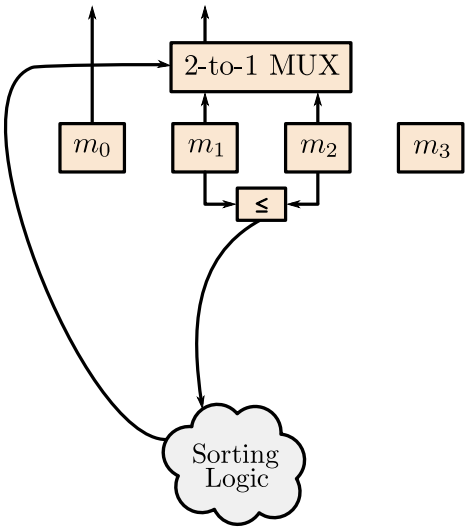

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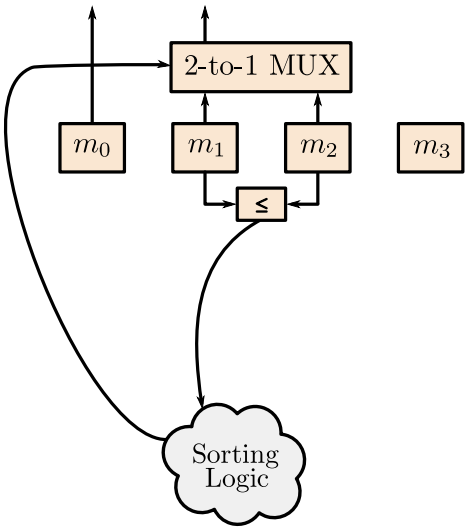

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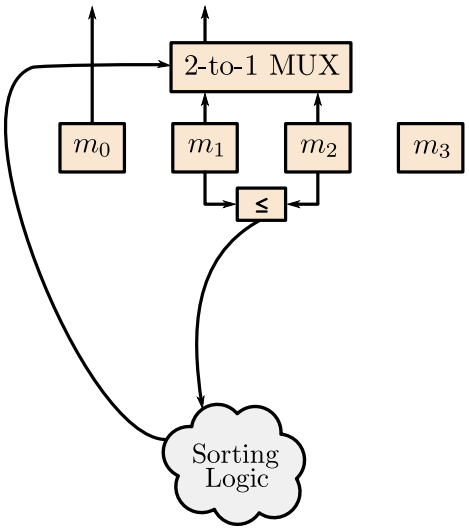

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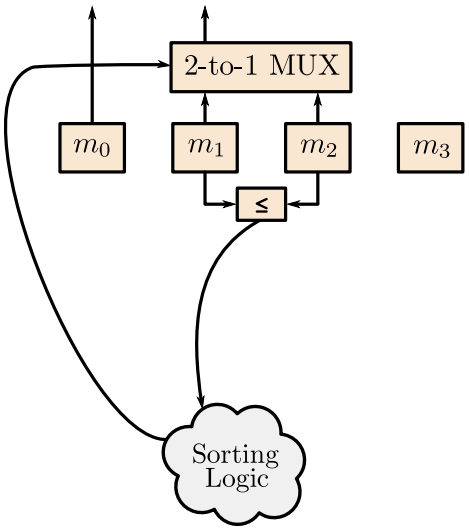

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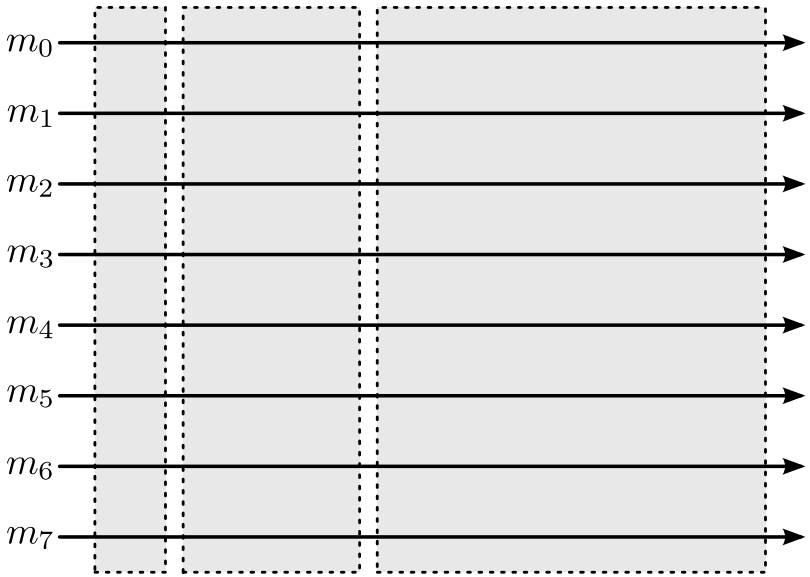

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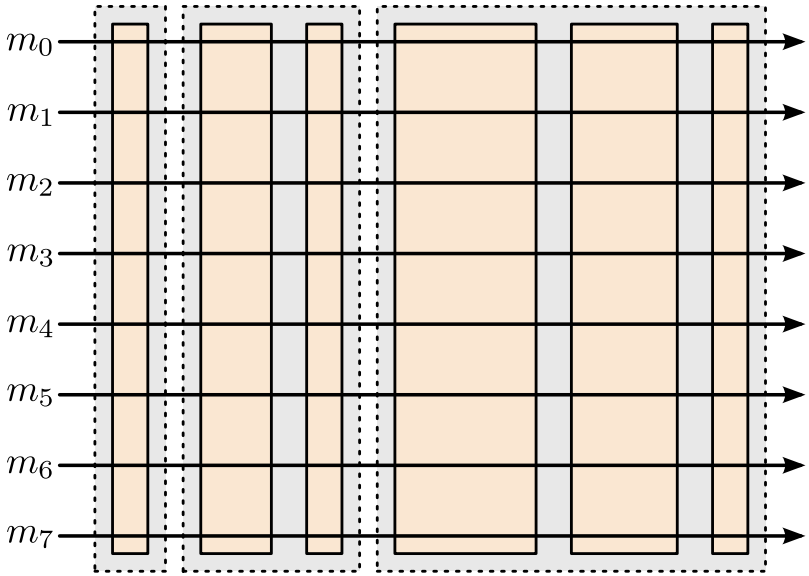

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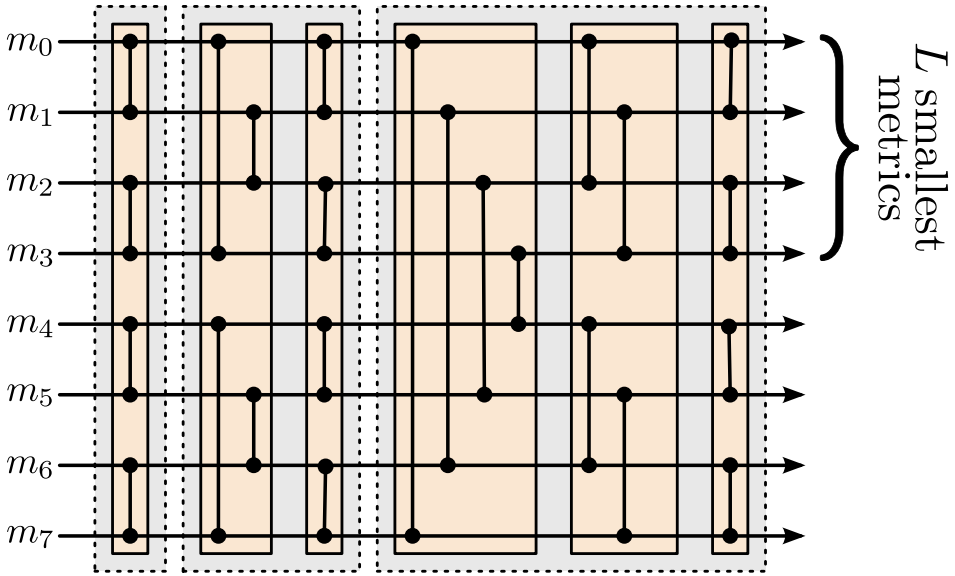

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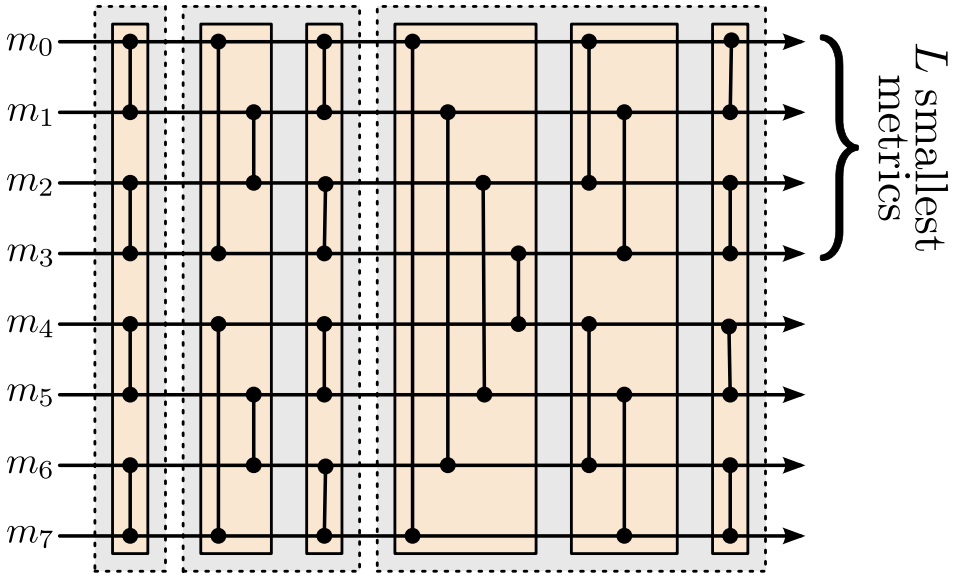

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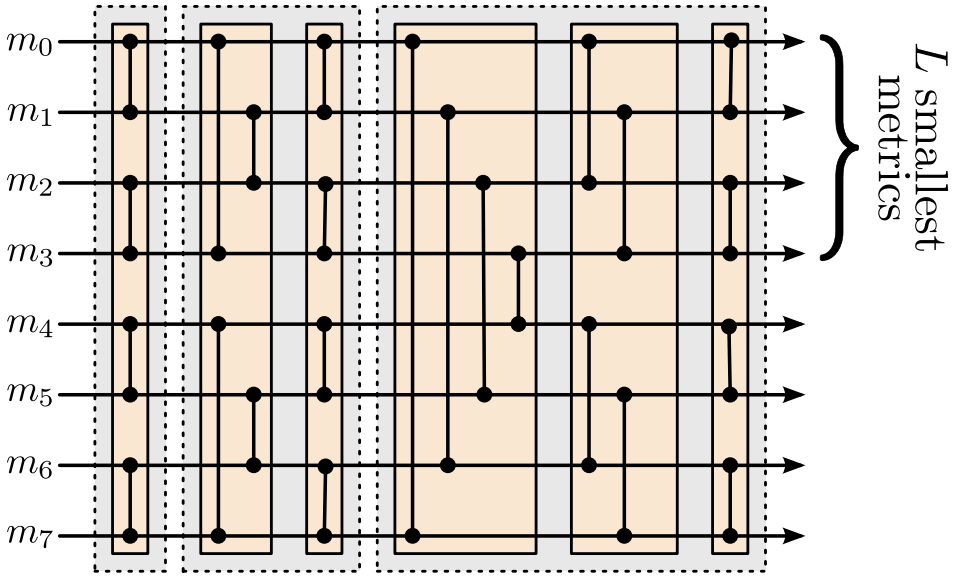

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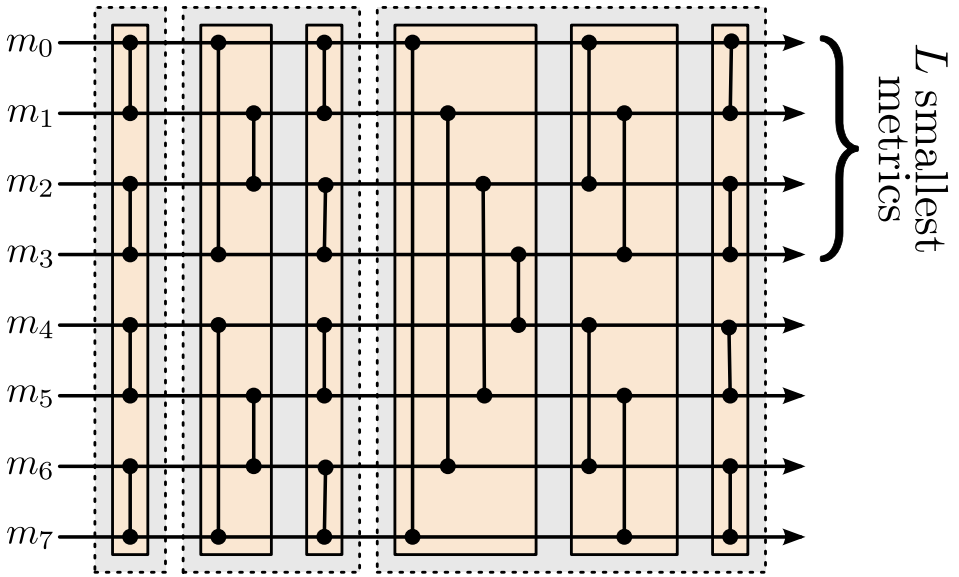

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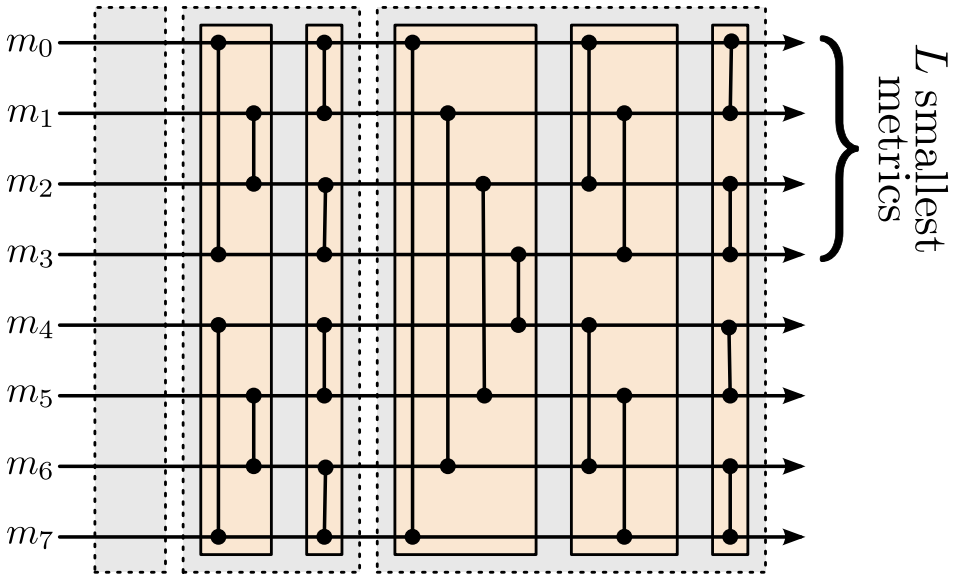

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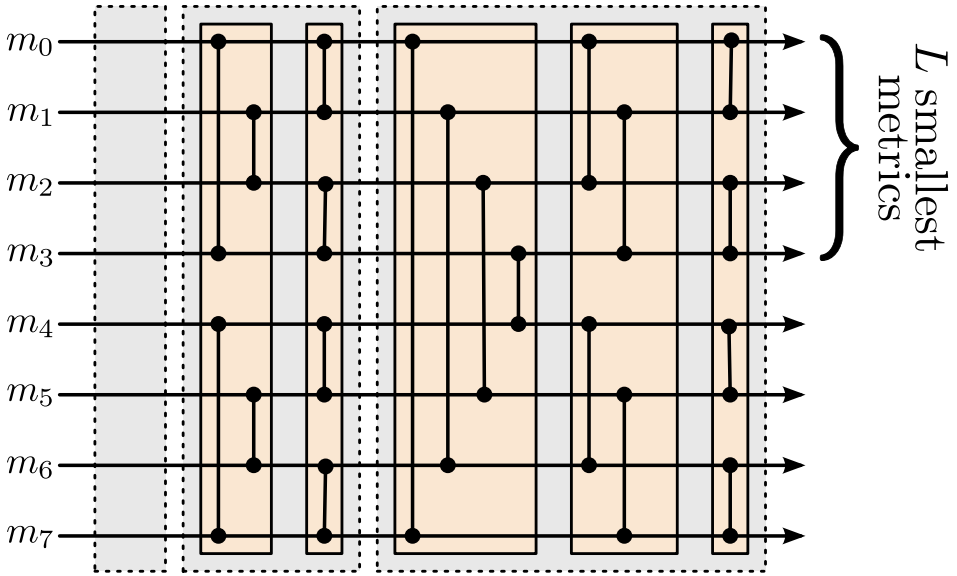

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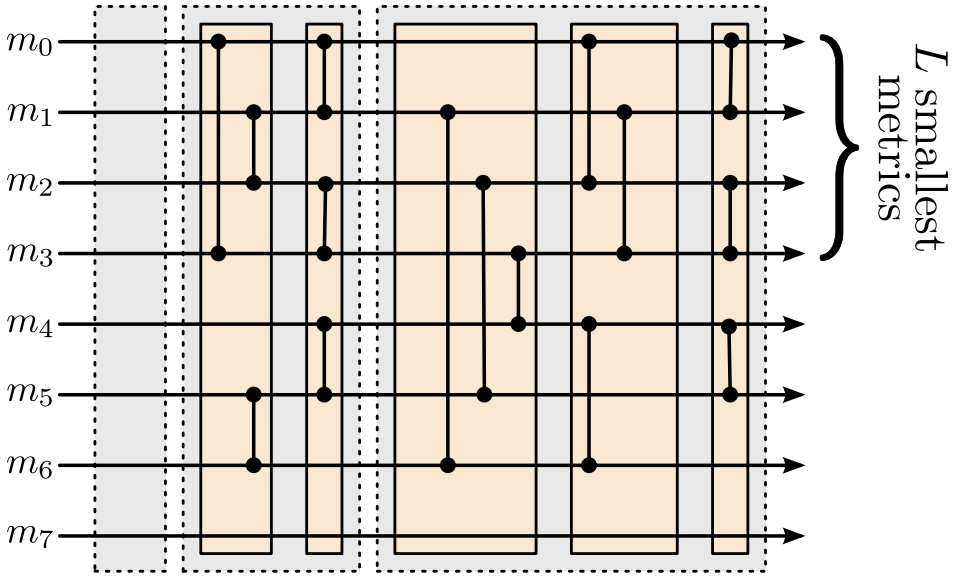

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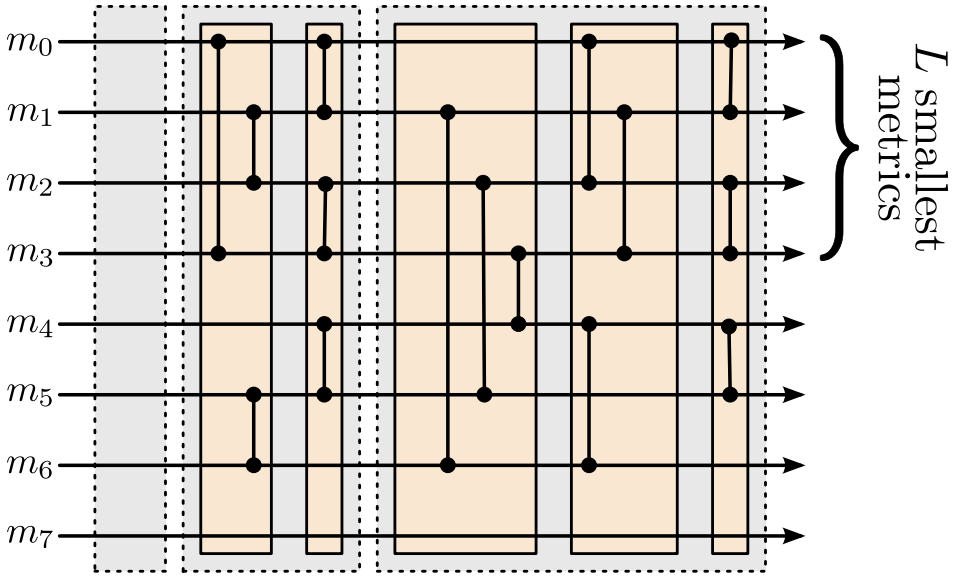

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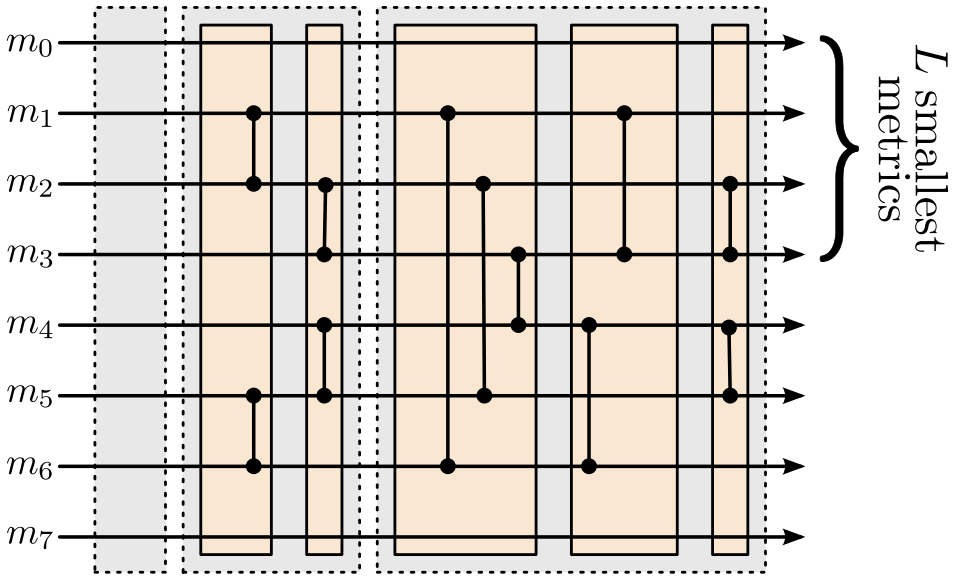

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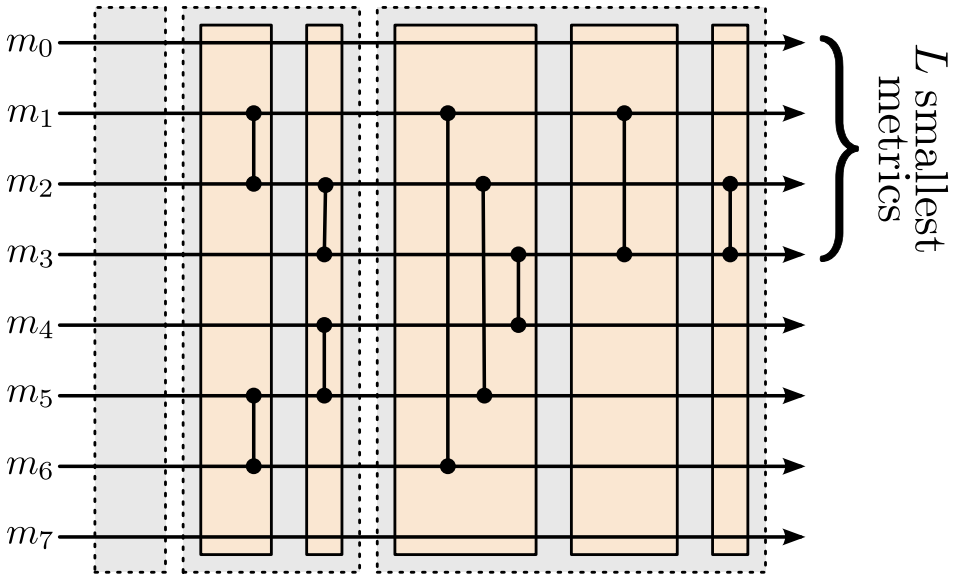

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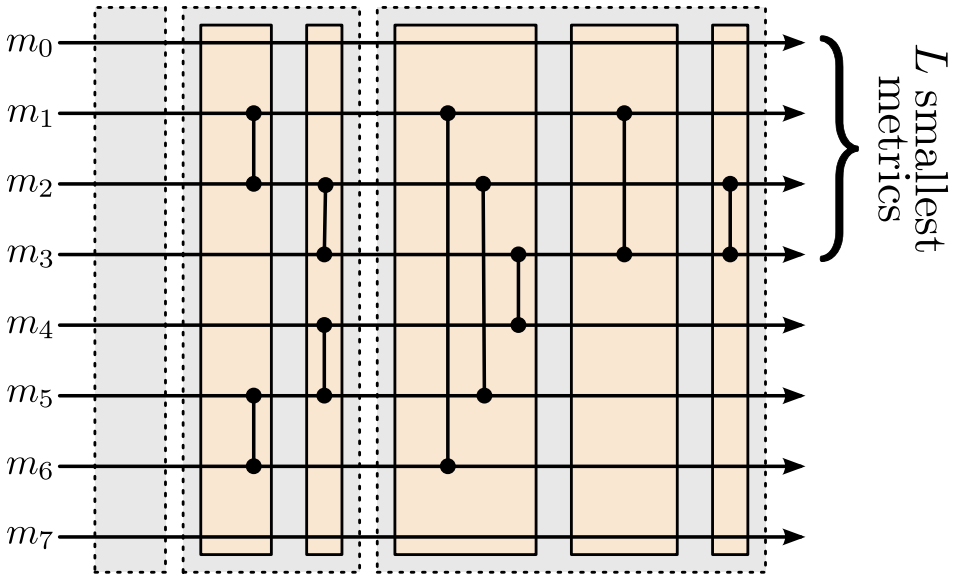

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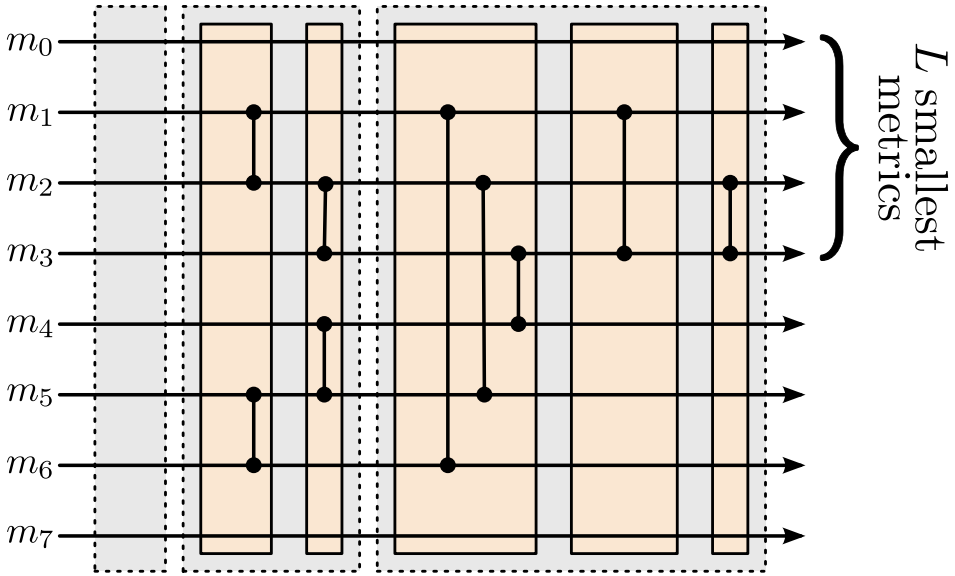

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- **Area** proportional to total CAS units: $(\frac{L}{2} - 1)(\log L)(\log L + 2) + 1$.

Operating frequency and area improvement **diminishes** as L is increased.

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$L = 2$	1370	2109	4545	608
$L = 4$	676	8745	952	3965
$L = 8$	347	27159	478	20748
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- **At least** 6% improvement in **maximum frequency**.
- **At least** 14% improvement in **area**.

Bubble Sorter

while *exists* ℓ *such that* $m_\ell > m_{\ell+1}$ **do**
    for $\ell = 2L - 1$ **to** 1 **do**
        if $m_{\ell-1} > m_\ell$ **then**
            Swap m_ℓ and $m_{\ell-1}$;

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Bubble Sorter

while *exists* ℓ *such that* $m_\ell > m_{\ell+1}$ **do**
    for $\ell = 2L-1$ **to** 1 **do** /* $\boxed{\ell = 7}$ */
        if $m_{\ell-1} > m_\ell$ **then**
            Swap m_ℓ and $m_{\ell-1}$;

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Bubble Sorter

while *exists* ℓ *such that* $m_\ell > m_{\ell+1}$ **do**
  for $\ell = 2L-1$ **to** 1 **do** /* $\ell = 6$ */
    if $m_{\ell-1} > m_\ell$ **then**
      Swap m_ℓ and $m_{\ell-1}$;

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    for $\ell = 2L-1$ **to** 1 **do** /* $\ell = 6$ */
        if $m_{\ell-1} > m_\ell$ **then**
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    if $m_{\ell-1} > m_\ell$ **then**
      Swap m_ℓ and $m_{\ell-1}$;

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while *exists* ℓ *such that* $m_\ell > m_{\ell+1}$ **do**
  for $\ell = 2L-1$ **to** 1 **do** /* $\ell = 4$ */
    if $m_{\ell-1} > m_\ell$ **then**
      Swap m_ℓ and $m_{\ell-1}$;

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Bubble Sorter

while *exists* ℓ *such that* $m_\ell > m_{\ell+1}$ **do**
    for $\ell = 2L-1$ **to** 1 **do** /* $\ell = 3$ */
        if $m_{\ell-1} > m_\ell$ **then**
            Swap m_ℓ and $m_{\ell-1}$;

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Bubble Sorter

while *exists* ℓ *such that* $m_\ell > m_{\ell+1}$ **do**
    for $\ell = 2L-1$ **to** 1 **do** /* $\ell = 2$ */
        if $m_{\ell-1} > m_\ell$ **then**
            Swap m_ℓ and $m_{\ell-1}$;

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- The execution of the **if** block depends only on the state of the list at the beginning of the **for** loop.
- Each element participates in at most one **swap** operation.
- The execution of the **for** loop can be parallelized.

Bubble Sorter

while *exists* ℓ *such that* $m_\ell > m_{\ell+1}$ **do**
  for $\ell = 2L-1$ **to** 1 **do** /* $\ell = 7\&3$ */
    if $m_{\ell-1} > m_\ell$ **then**
      Swap m_ℓ and $m_{\ell-1}$;

Example $(3, 11, 9, 17, 10, 20, 18, 19)$

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- Each element participates in at most one **swap** operation.
- The execution of the **for** loop can be parallelized.

Bubble Sorter

while *exists* ℓ *such that* $m_\ell > m_{\ell+1}$ **do**
  for $\ell = 2L-1$ **to** 1 **do** /* $\ell = 7\&3$ */
    if $m_{\ell-1} > m_\ell$ **then**
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while *exists* ℓ *such that* $m_\ell > m_{\ell+1}$ **do**
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- At odd (even) rounds of the **while** loop, CAS operations only take place for even (odd) ℓ.

Hardware Implementation of Bubble Sort

- $2L-2$ stages implementing (at most) $2L-2$ iterations of the **while** loop.

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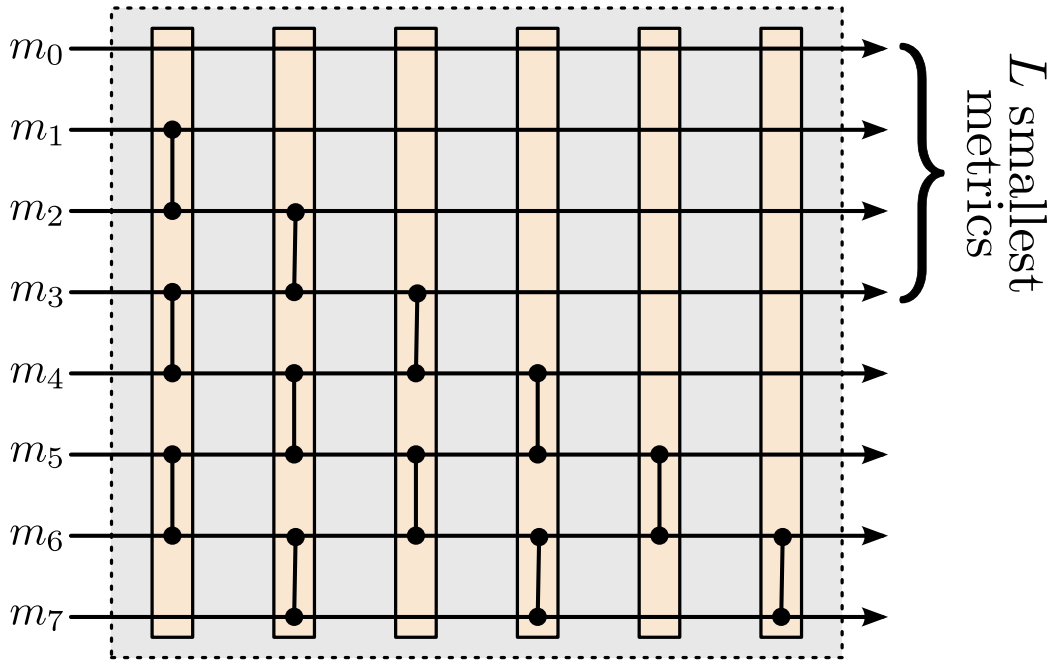

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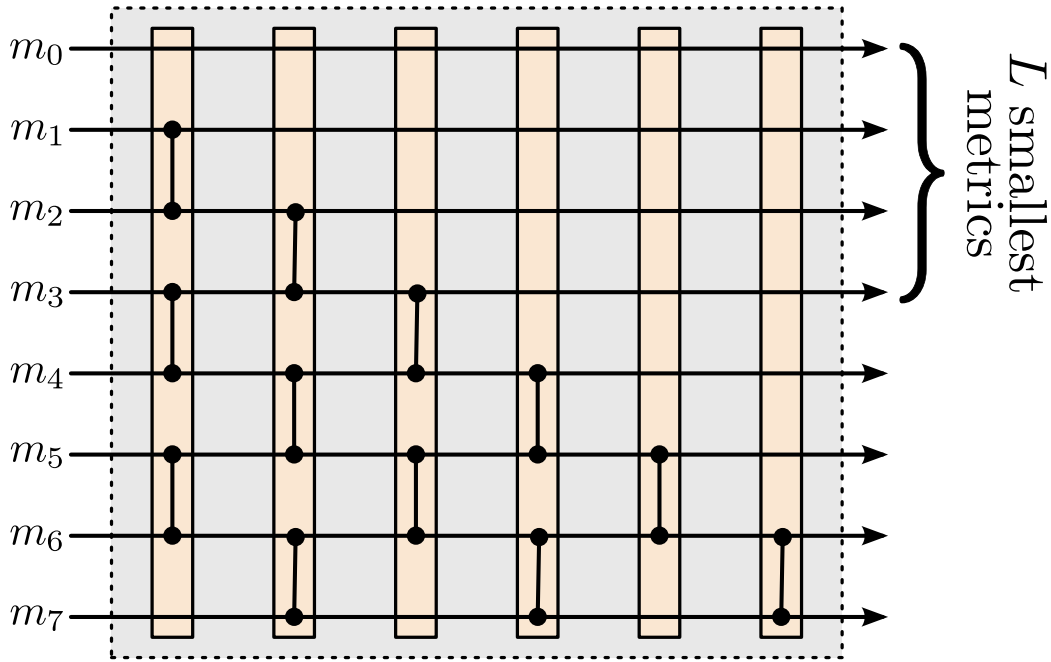

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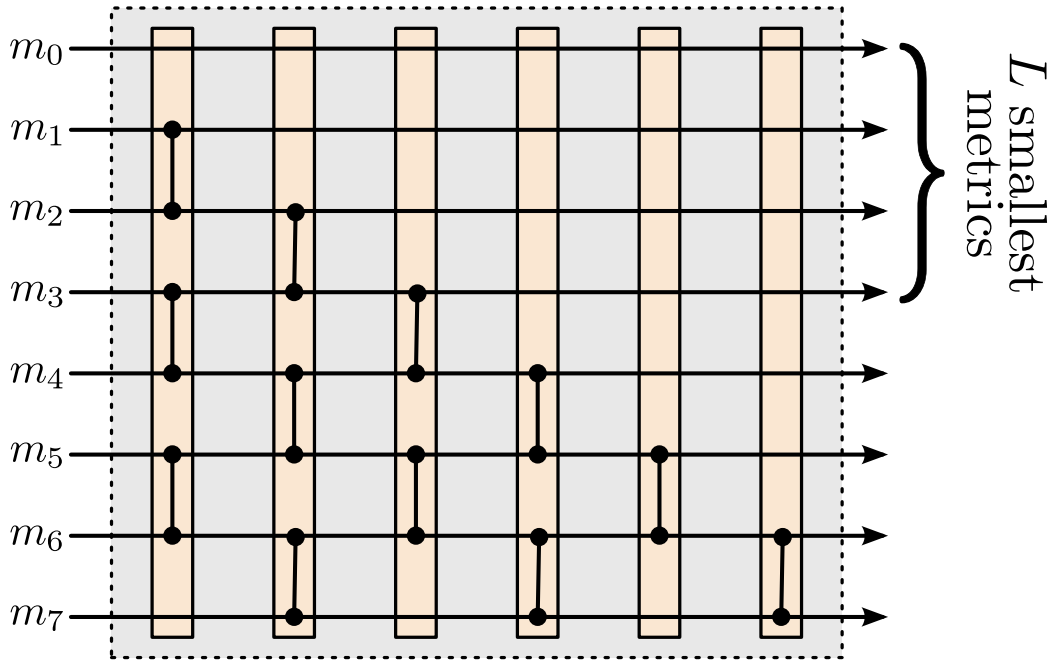

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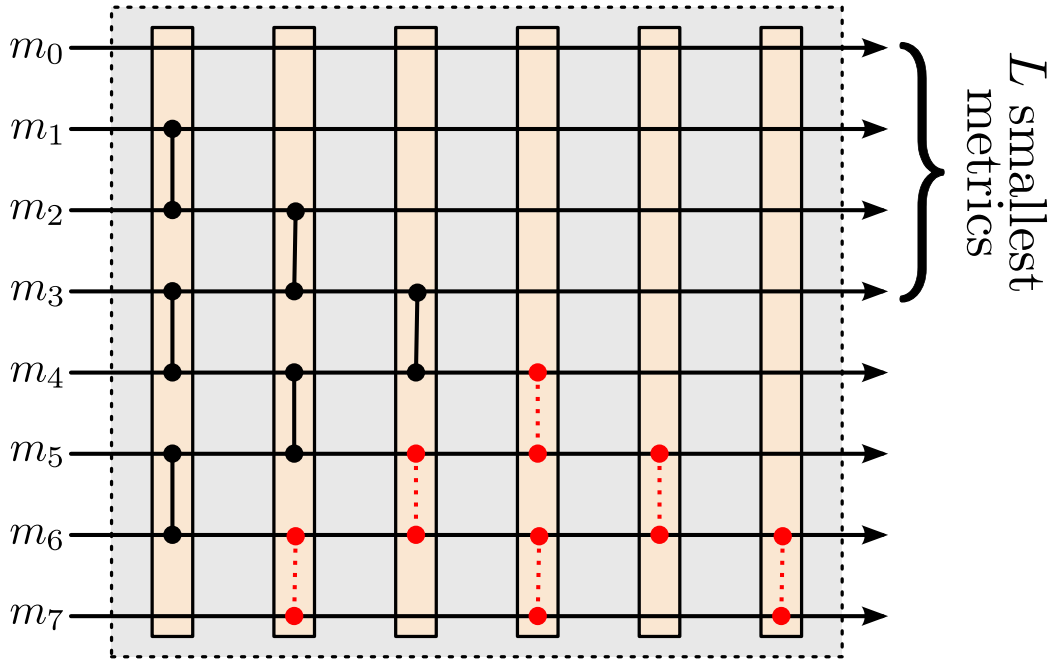

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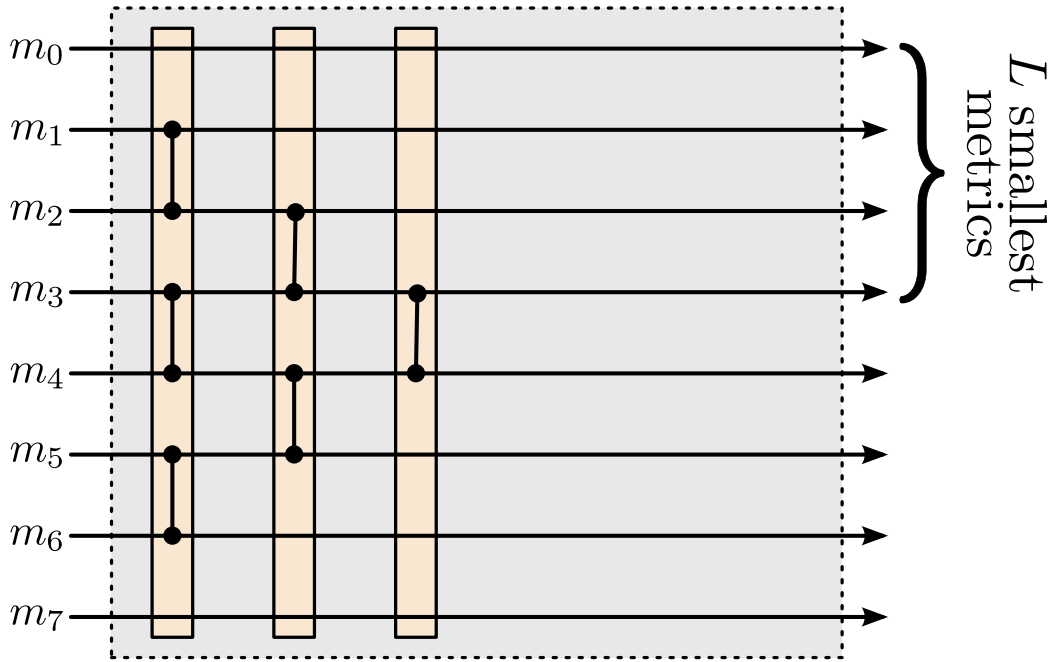

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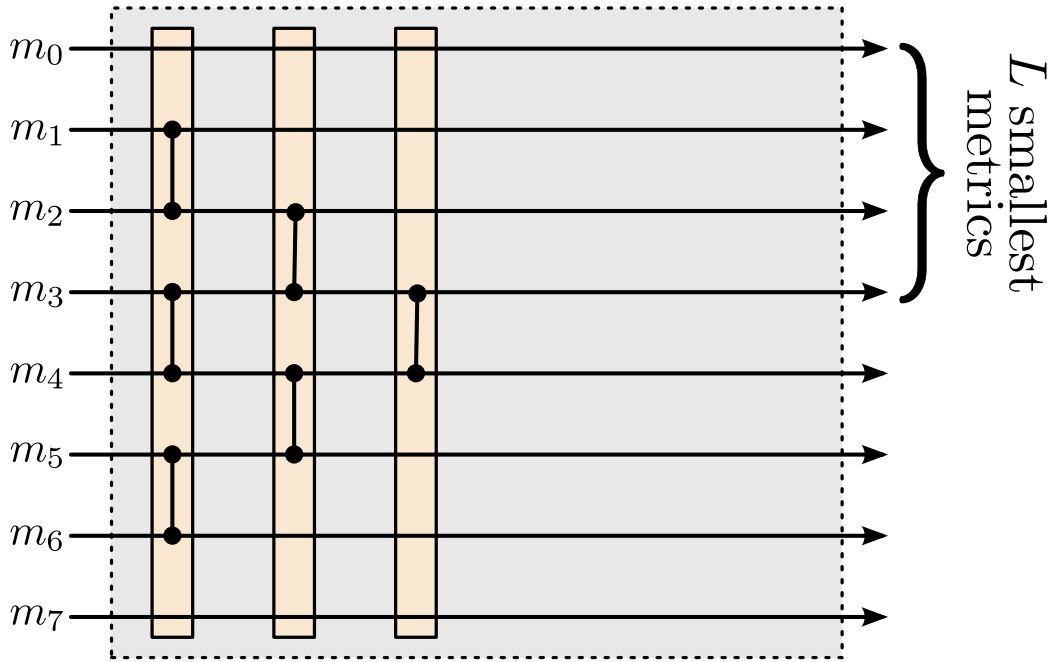

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$\Rightarrow \frac{1}{2}L(L-1)$ CAS units in $L-1$ stages.

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$L = 4$	2083	3703	952	3965	1388	2756
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Thank you!

Questions?

VHDL code for all sorters available at: `http://tcl.epfl.ch`